

Invariants of the Kerr Vacuum

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The Kerr vacuum has two independent invariants derivable from the Riemann tensor without differentiation. Both of these invariants must be examined in order to avoid an erroneous conclusion that the ring singularity of this spacetime is “directional”.

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Recently Schmidt [1] has reminded us, by way of a simple example, that the square of the Weyl tensor can be negative. This behavior is a well known property of the Kerr vacuum [2], [3], [4] in which the Weyl tensor becomes highly anisotropic sufficiently close to the singularity. Indeed, the divergence of this scalar (equivalent here to the Kretschmann scalar) is a standard way in which texts point out the singularity in this spacetime (e.g. [5], [6], [7]) though an explicit form of this scalar is a little hard to find in the texts (correct forms are give in [8] and [9] and an incorrect form is given in [10]). The scalar is not difficult to calculate by hand (using, for example, the Newman-Penrose formalism) and so it not a surprise that it was trivial to calculate (in a radiating Kerr-Newman generalization) on a personal computer a decade ago [11]. The Kerr vacuum has two independent invariants derivable from the Riemann tensor without differentiation. In this note we examine each in detail and point out that both must be examined in order to avoid an erroneous conclusion that the ring singularity is in any sense “directional”.

In terms of the familiar Boyer-Lindquist coordinates [12], writing $x \equiv r/a$, $a \neq 0$ and $W \equiv C_{ijkl}C^{ijkl}$ where C_{ijkl} is the Weyl tensor, it follows that

$$\mathcal{W} \equiv \frac{W a^6}{48 m^2} = \frac{(x - x1)(x + x1)(x - x2)(x + x2)(x - x3)(x + x3)}{(x^2 + x1^2)^6} \quad (1)$$

where $x1 \equiv \cos(\theta)$, $x2 \equiv (2 + \sqrt{3}) \cos(\theta)$ and $x3 \equiv (2 - \sqrt{3}) \cos(\theta)$. Note that (1) holds for all $a \neq 0$ and $m \neq 0$. Along $x = 0$, $\mathcal{W} \cos(\theta)^6 = -1$ but along $\theta = \pi/2$, $\mathcal{W} x^6 = +1$ and so \mathcal{W} fails to be continuous in the limit $x \rightarrow 0$, $\theta \rightarrow \pi/2$ ($\equiv \mathcal{S}$). Indeed \mathcal{W} need not diverge in the limit \mathcal{S} along trajectories asymptotic to $x = \pm x1, \pm x2, \pm x3$. The sign of \mathcal{W} is summarized in Figure 1 and some level curves of constant \mathcal{W} are shown in Figure 2, both in the $\theta - x$ plane. The function \mathcal{W} is plotted in Figures 3 and 4. This extends the analysis in [2] and in [4] (the special case given in Figure 2 of [4] is incomplete).

Any vacuum solution of Einstein's equations has only one further independent quadratic invariant, ${}^*W \equiv {}^*C_{ijkl}C^{ijkl}$ where ${}^*C_{ijkl}$ is dual to the Weyl tensor. (See, for example, [13]. W and *W are equivalent to the real and imaginary parts of the complex Weyl scalar in spinor notation. In [3] *W is called the Chern-Pontryagin invariant.) With x and $x1$ defined as in (1) it follows that

$${}^*\mathcal{W} \equiv \frac{{}^*W a^6}{96 m^2} = \frac{-3 x x1 (x - x4)(x + x4)(x - x5)(x + x5)}{(x^2 + x1^2)^6} \quad (2)$$

where $x4 \equiv \sqrt{3} \cos(\theta)$ and $x5 \equiv \cos(\theta)/\sqrt{3}$. There are no further independent (non-differential) invariants in the Kerr vacuum (as it is of Petrov type D). Again note that (2) holds for all $a \neq 0$ and $m \neq 0$. Now ${}^*\mathcal{W} = 0$ along $x = 0$ but, for example, $16 {}^*\mathcal{W} \cos(\theta)^6 = +1$ along $x = x1$ and so ${}^*\mathcal{W}$ also fails to be continuous in the limit \mathcal{S} . ${}^*\mathcal{W} = 0$ along $x = 0$ and $\theta = \pi/2$ and need not diverge in the limit \mathcal{S} along trajectories asymptotic to $x = \pm x4$ and $\pm x5$. The sign of ${}^*\mathcal{W}$ is summarized in Figure 5 and some level curves of constant ${}^*\mathcal{W}$ are shown in Figure 6, again both in the $\theta - x$ plane. The function ${}^*\mathcal{W}$ is plotted in Figures 7 and 8.

Whereas there exist trajectories (albeit non-geodesic) along which \mathcal{W} or ${}^*\mathcal{W}$ do not diverge in the limit \mathcal{S} , there exists no trajectory along which both \mathcal{W} and ${}^*\mathcal{W}$ remain finite in this limit. In this sense the singularity of the Kerr vacuum is not directional.

Dynamic images of \mathcal{W} and ${}^*\mathcal{W}$ are available at <http://grtensor.org/negweyl> the details of which are difficult to summarize in static images.

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 - [14] This is a package which runs within Maple. It is entirely distinct from packages distributed with Maple and must be obtained independently. The GRTensorII software and documentation is distributed freely on the World-Wide-Web from the address <http://grtensor.org>

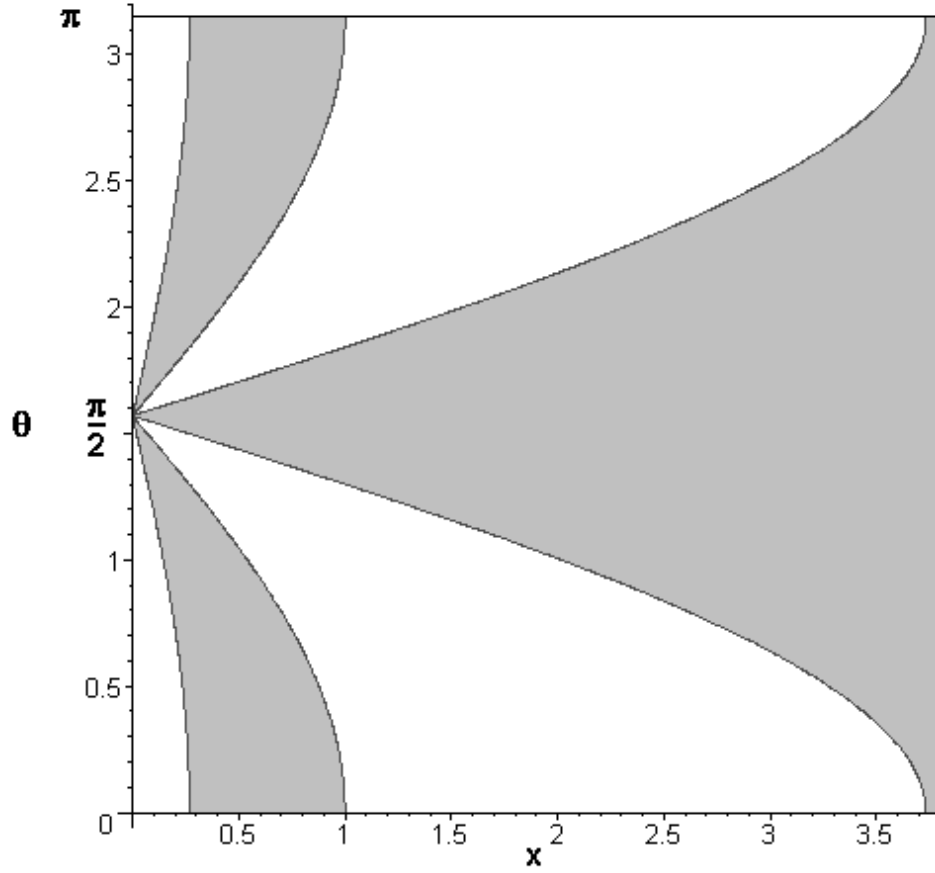


FIG. 1: Regions with $\mathcal{W}(\equiv \frac{C_{ijkl}C^{ijkl}a^6}{48m^2}) > 0$ are shaded. The shaded region extends indefinitely to the right where $\mathcal{W} \rightarrow 0$ as $x \rightarrow \infty$. The boundaries $\mathcal{W} = 0$ are given by $x = \pm x_1, \pm x_2, \pm x_3$ as explained in the text. (In the corresponding Kerr-Schild representation the boundaries reduce to circles, two of radius 2 centered on $\pm\sqrt{3}$ and one of radius 1 centered at the origin. For our purposes here we find the $\theta - x$ plane more useful.)

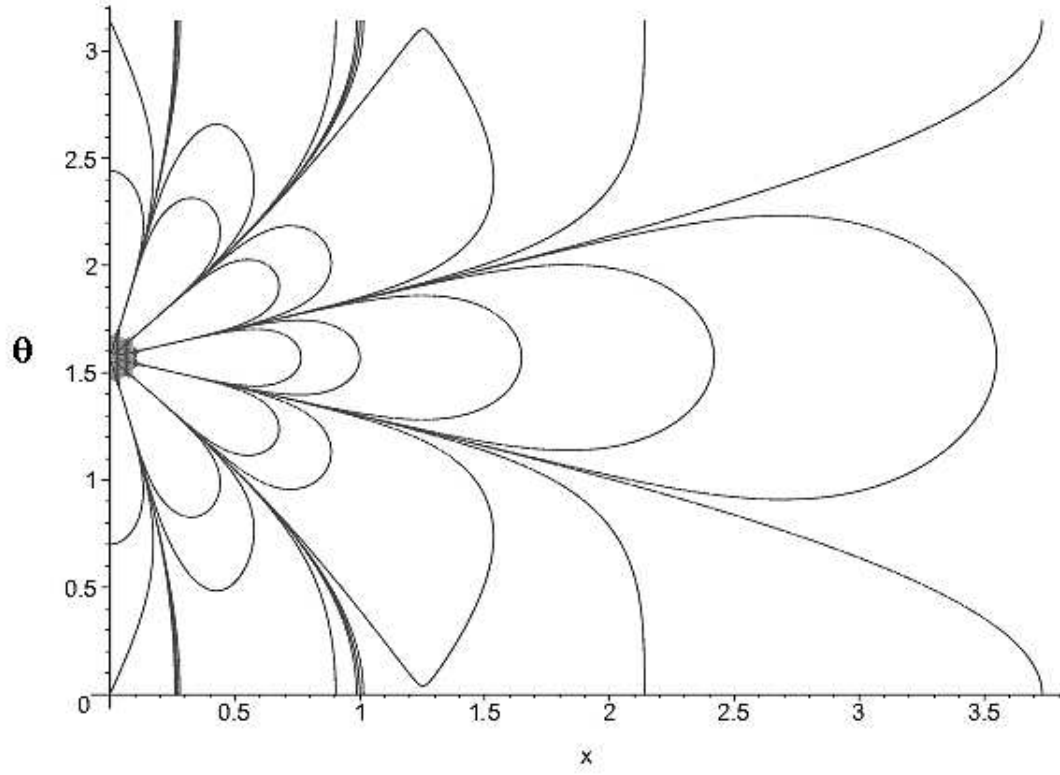


FIG. 2: Level curves of constant \mathcal{W} . The values of \mathcal{W} shown are 5, 1, 0.05, 0.005, 0.0005, 0, -0.005, -0.0368, -1 and -5.

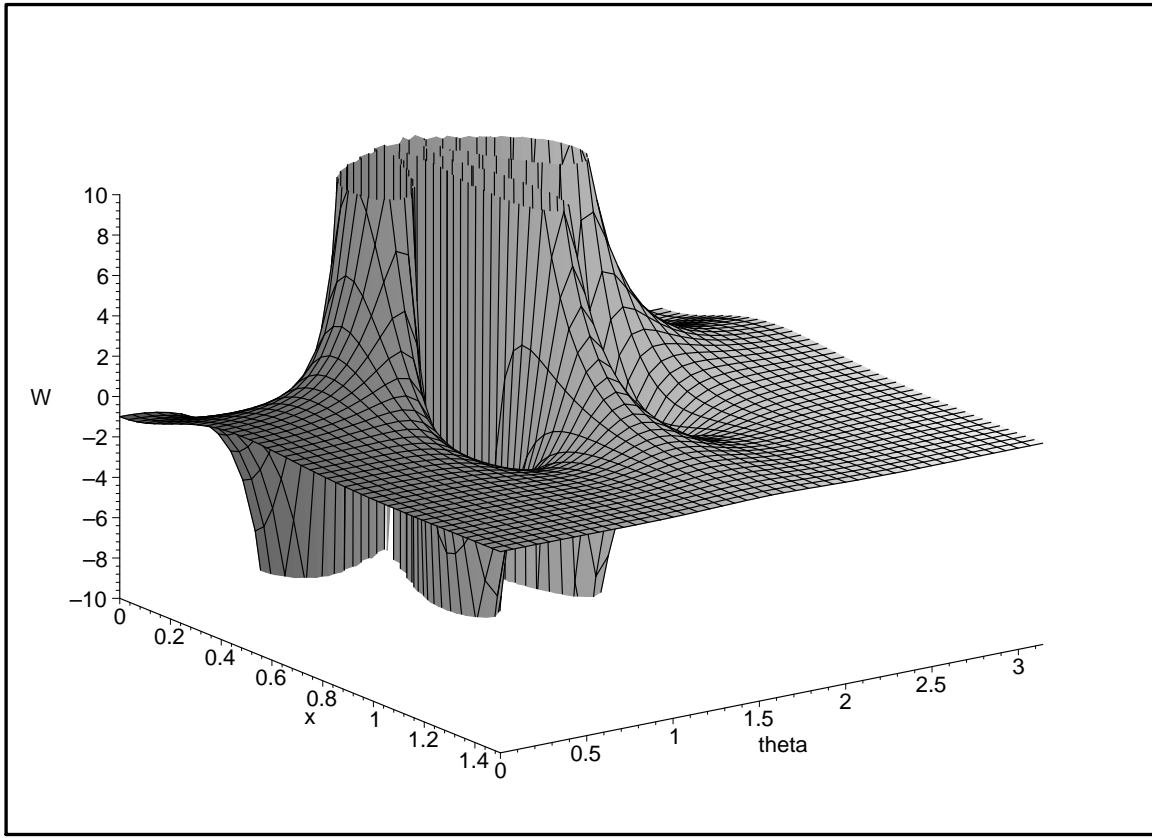


FIG. 3: Plot of $\mathcal{W}(\equiv \frac{C_{ijkl}C^{ijkl}a^6}{48m^2})$ showing a variation in both $x \equiv \frac{r}{a}$ and θ .

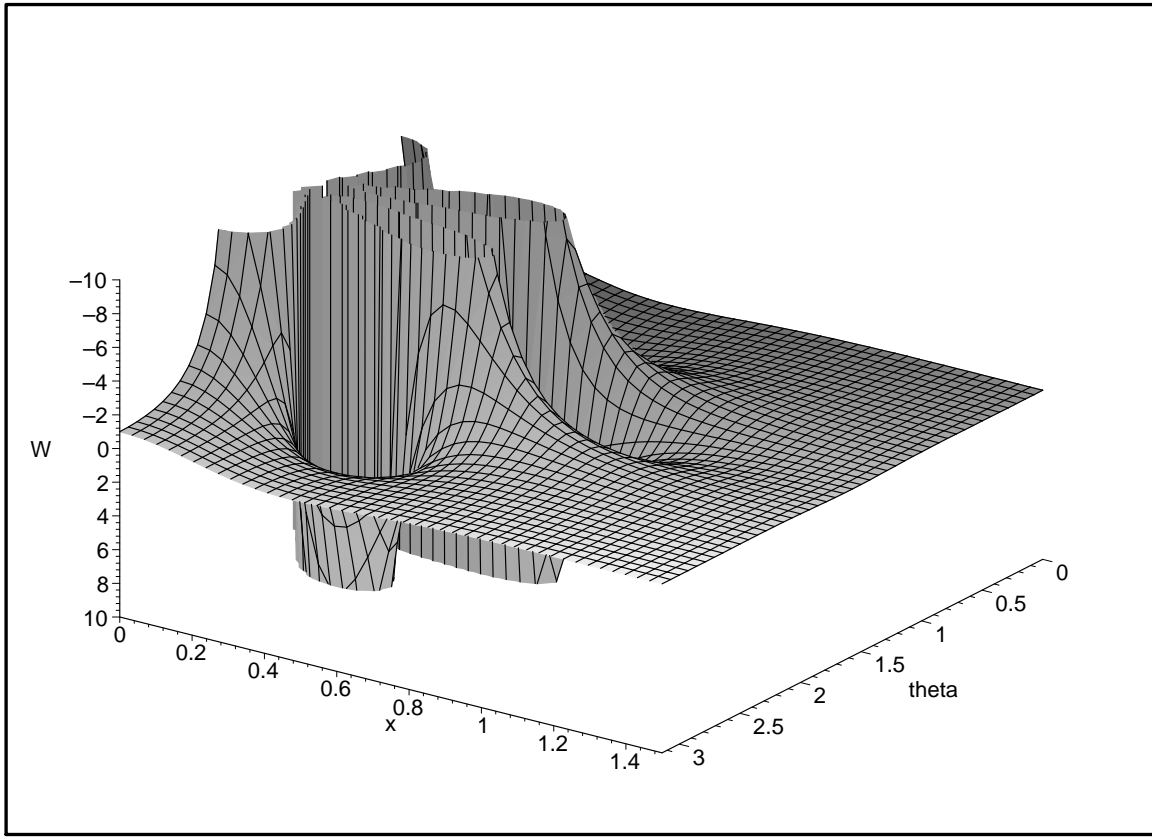


FIG. 4: Plot of $\mathcal{W}(\equiv \frac{C_{ijkl}C^{ijkl}a^6}{48m^2})$ showing a variation in both $x \equiv \frac{r}{a}$ and θ . This is the underside of Figure 3.

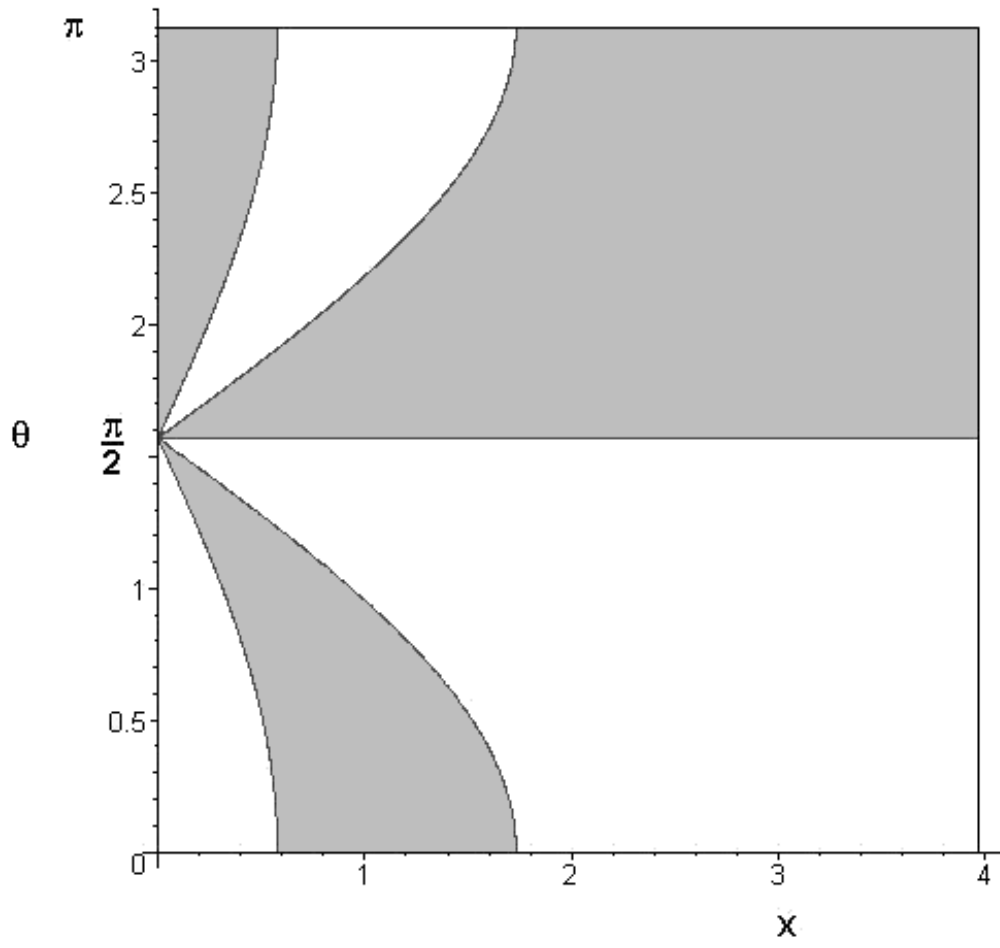


FIG. 5: Regions with ${}^*\mathcal{W}(\equiv \frac{{}^*C_{ijkl}C^{ijkl}a^6}{96m^2}) > 0$ are shaded. The boundaries ${}^*\mathcal{W} = 0$ are given by $x = 0$, $\theta = \pi/2$, $x = \pm x_4$ and $x = \pm x_5$ as explained in the text.

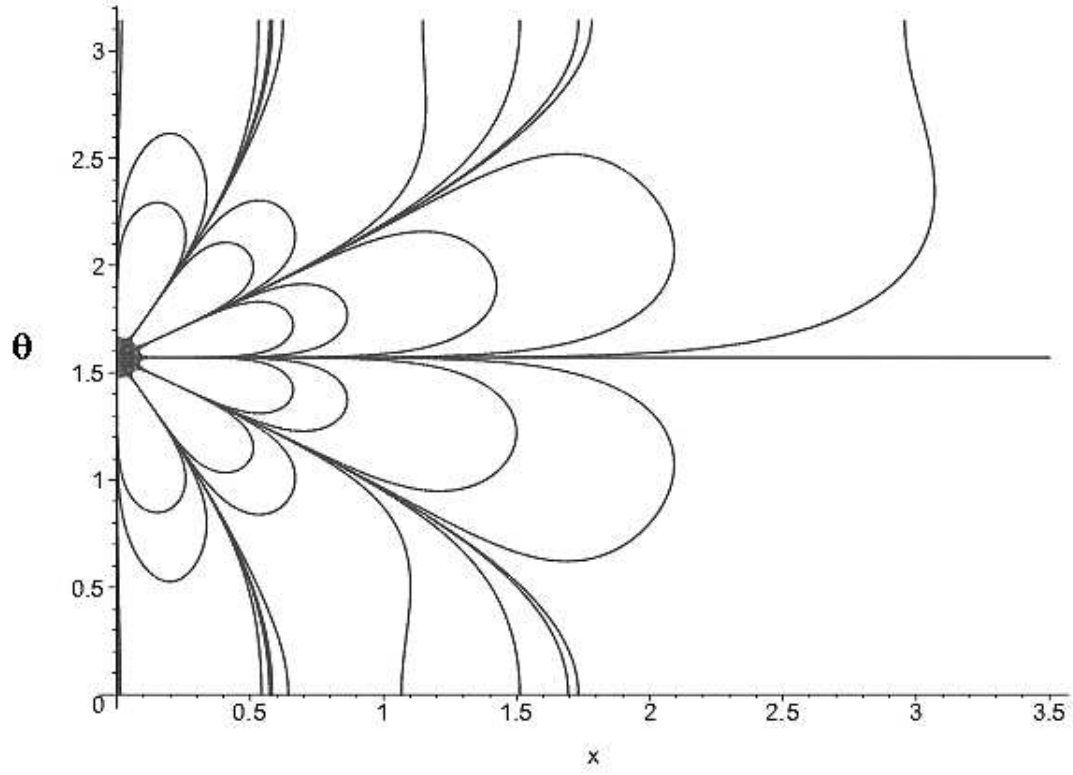


FIG. 6: Level curves of constant $^*\mathcal{W}$. The values of $^*\mathcal{W}$ shown are 5, 1, 0.05, 0.005, 0.0005, 0, -0.005, -0.0368, -1 and -5.

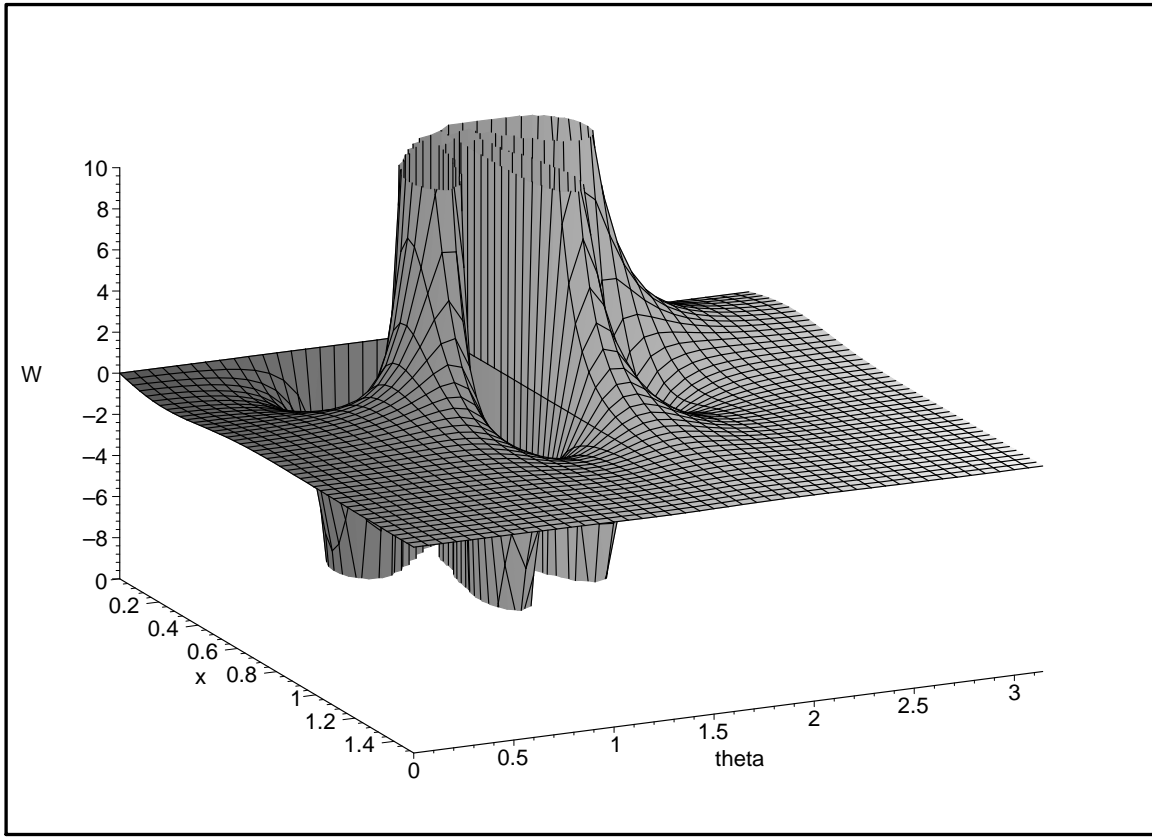


FIG. 7: Plot of ${}^*\mathcal{W}(\equiv \frac{{}^*C_{ijkl}C^{ijkl}a^6}{96m^2})$ showing a variation in both $x \equiv \frac{r}{a}$ and θ .

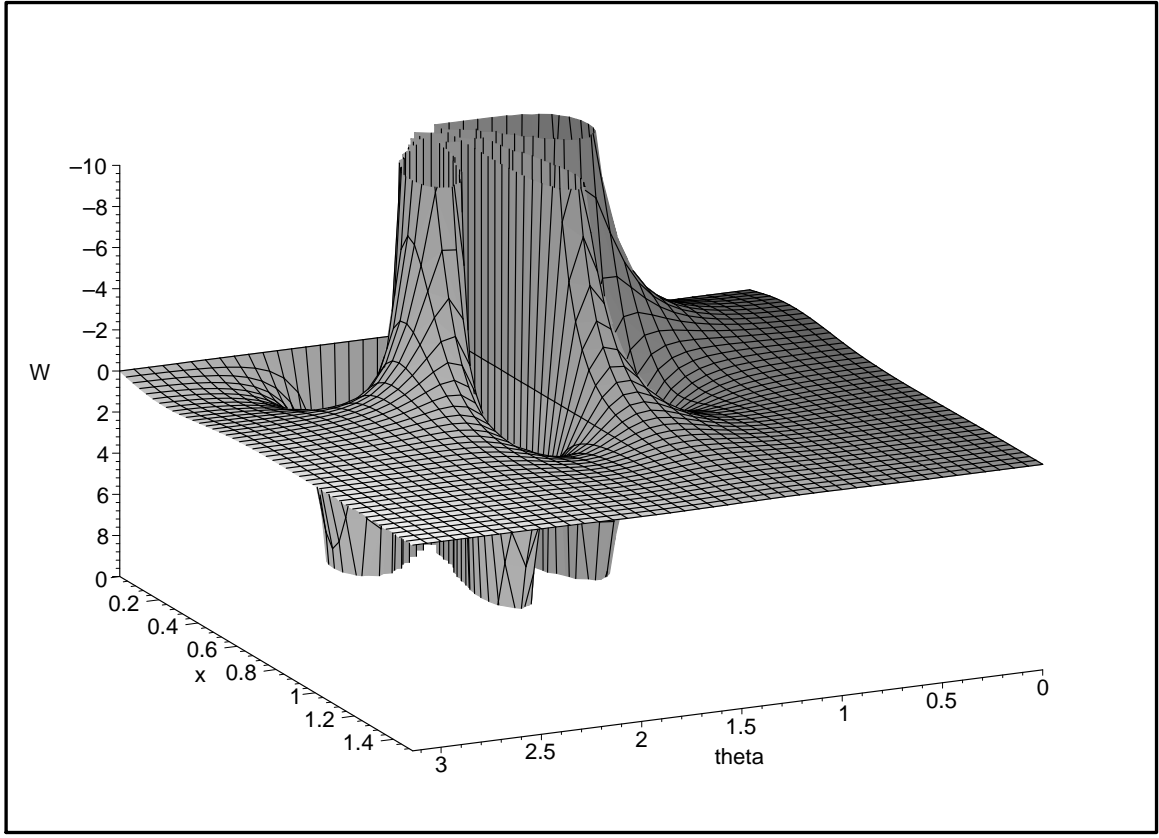


FIG. 8: Plot of $\mathcal{W}(\equiv \frac{{}^*C_{ijkl}C^{ijkl}a^6}{96m^2})$ showing a variation in both $x \equiv \frac{r}{a}$ and θ . This is the underside of Figure 7.